# Limit Cycle Analysis Applied to the Oscillations of Decelerating Blunt-Body Entry Vehicles

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***ABSTRACT***

*Many blunt-body entry vehicles have nonlinear dynamic stability characteristics that produce self-limiting oscillations in flight. Several different test techniques can be used to extract dynamic aerodynamic coefficients to predict this oscillatory behavior for planetary entry mission design and analysis. Some of these test techniques impose boundary conditions that alter the oscillatory behavior from that seen in flight. The test conditions of three commonly used dynamic test techniques are assessed to highlight these effects. Analytical solutions to the constant-coefficient planar equations-of-motion for each case are developed to show how the same blunt body behaves differently depending on the imposed test conditions. The energy equation is applied to further illustrate the forces and moments that govern the observed dynamics. Then, the mean value theorem is applied to the energy rate equation to find the effective damping for an example blunt body with nonlinear, self-limiting dynamic characteristics. This approach is used to predict constant-energy oscillatory behavior and the equilibrium oscillation amplitudes for the various test conditions. These predictions are verified with planar simulations. The analysis presented provides an overview of dynamic stability test techniques and illustrates the effects of dynamic stability, static aerodynamics and test conditions on observed dynamic motions. It is proposed that these effects may be leveraged to develop new test techniques and refine test matrices in future tests to better define the nonlinear functional forms of blunt body dynamic stability curves.*

## NOMENCLATURE

| | | | |
|---|---|---|---|
| $A$ | Angle-of-attack constant | $R$ | Earth radius |
| $C_A$ | Axial force coefficient | $S$ | Reference area |
| $C_D$ | Drag force coefficient | $T$ | Period of oscillation |
| $C_L$ | Lift force coefficient | $t$ | Time |
| $C_{L_\alpha}$ | Lift-curve slope | $V$ | Velocity |
| $\overline{C_{L_\alpha} + C_{m_q}}$ | Effective damping with lift effects | Greek | |
| $C_{m_\alpha}$ | Pitching moment slope | $\alpha$ | Angle-of-attack |
| $C_{m_q} + C_{m_{\dot\alpha}}$ | Pitch damping coefficient | $\gamma$ | Flight-path angle |
| $\overline{C_{m_q}}$ | Effective pitch damping | $\delta$ | Phase shift constant |
| $C_N$ | Normal force coefficient | $\mu$ | Euler-Cauchy damping constant |
| $d$ | Reference diameter | $\nu$ | Euler-Cauchy frequency constant |
| $f$ | Unspecified function | $\xi_{1,2}$ | Constant velocity damping coeffs. |
| $g$ | Gravitational acceleration | $\rho$ | Density |
| $I$ | Moment-of-inertia | $\omega$ | Oscillation frequency |
| $K$ | Oscillatory energy | Subscripts | |
| $M$ | Mach number | $i, f$ | Initial and final conditions |
| $m$ | Mass | $o$ | Function of oscillation amplitude |

## 1.0 INTRODUCTION

Blunt bodies have been used for decades to protect and decelerate robotic payloads through atmospheres on other planets and to safely return human and robotic payloads to earth. Various sphere-cone and spherical-section forebodies have proven to be efficient designs for decelerating payloads and astronauts from very large entry speeds, while minimizing the peak heating rate and heat load the vehicle must withstand. However, one adverse aerodynamic property associated with many blunt vehicles is a bounded dynamic instability that tends to start near Mach 3.5, becoming more unstable with decreasing Mach number. This instability can cause blunt-body oscillations to grow so much that a safe parachute deployment is not possible. In extreme cases, dynamic instability can result in a capsule tumbling if not assisted by a drogue parachute or other stabilizing device.

The pitch damping characteristics of blunt bodies tend to be nonlinear. For non-lifting vehicles (e.g. axisymmetric with no radial cg offset) the peak pitch damping coefficient (most unstable) typically occurs at $\alpha = 0°$, dropping off and becoming negative at larger angles-of-attack [1–3]. For constant velocity flight, a statically stable blunt body oscillates and reaches equilibrium at an oscillation amplitude where undamping dynamic moments at small angles are balanced by the damping moments at large angles-of-attack. This is true for full-scale vehicles and also occurs during ground-based ballistic range testing. As the vehicle decelerates, dynamic pressure drops and static pitching moments on the vehicle are reduced. This relaxation of the static stability "spring stiffness" causes a decrease in frequency and increase in oscillation amplitude. Therefore a blunt body reaching oscillation amplitude equilibrium while traveling at constant velocity will exhibit amplitude growth in decelerating flight.

Capsule forebody shapes are typically driven by heating and static stability constraints. Packaging and mass properties requirements tend to drive the backshell shapes. Therefore backshell geometries are as varied as the missions for which they are used. The backshell geometry is thought to be the dominant factor effecting how the wake structure interacts with the body and ultimately the level of dynamic stability of the vehicle. There are ongoing efforts to predict blunt-body dynamic stability using computational methods [4, 5], but to date, no practical predictive capability exists. Experimental methods remain a fundamental component in the assessment of blunt body dynamic stability. Ballistic range testing is a frequently used technique. Free flight data in a ballistic range is not corrupted by a sting disrupting the wake flow. Also, when properly scaled, the flight dynamics are essentially the same as seen in full-scale flight, whereas wind tunnels are typically limited to oscillations about a fixed point. One exception is testing in a vertical spin tunnel, where the capsule is again flying freely with the weight of the model in equilibrium with the drag acting upon it. The flight-like motions achieved in the ballistic range or spin tunnel has advantages and disadvantages. While these techniques give a good representation of the motions expected in flight, the dynamic moments driving the motions are difficult to quantify in coefficient form for use in simulation, as no direct measurement of the moments acting on the vehicle are possible.

The nonlinear characteristics of blunt body dynamics, and the difficulty in modeling the governing flow physics, coupled with the amplitude changes due to deceleration, combine to obfuscate the interchange of forces and moments on the capsule as it moves along a trajectory. Limit-cycle analysis helps explicate the nonlinear pitch damping characteristics of blunt capsules, but velocity changes make limit-cycle analysis more complicated. This work attempts to separate the capsule aerodynamic characteristics from the trajectory effects and test conditions to interpret ballistic range results and the role of dynamic stability in the overall stability of a blunt capsule in flight. The effects of dynamic stability on oscillations in three scenarios are provided: constant velocity, free-to-oscillate; constant velocity, free-to-heave and oscillate; and decelerating, free-to-heave and oscillate. The oscillation energy equation and analytic solutions are used to show how pitch damping characteristics, the lift-curve slope and other factors influence the flight dynamics in these scenarios .

This paper will provide an overview of blunt body flight characteristics for those new to blunt body aerodynamics and demonstrate the effects of different test conditions on the observed dynamics. Others have looked at blunt body limit cycles in some detail. For example, Chapman and Yates [6] give an excellent explanation of limit cycle analysis as applied to blunt body aerodynamics. The work presented here takes a step back from that analysis and uses more simplified equations of motion to identify and compare the first-order effects on blunt body dynamics. Analytic solutions and numerical simulations are used to illustrate, to those new to dynamic instabilities and aerodynamic limit cycles, the contributions of dynamic and static aerodynamics and freestream conditions to the oscillatory behavior of blunt bodies and the addition or removal of oscillatory energy. Even among aerospace engineers who work with entry capsules, the interplay of translational motion, unsteady aerodynamic forces and oscillatory motion is not always appreciated at a fundamental level. This paper attempts to provide a framework to help the engineer understand the roles of different forces and moments acting on a blunt body that result in limit-cycle and near limit-cycle motions.

## 2.0 TEST TECHNIQUES AND APPLICATIONS

To understand the examples presented in later sections, a brief overview of commonly used dynamic aerodynamic test techniques is presented. Within each type are variations which can effect the observed capsule motions and each technique has benefits and limitations. Variations of two major types of testing, wind tunnel and ballistic range, are described and then contrasted with planetary entry trajectories. From these different test techniques, three cases will be used for comparison by analytic solution in the following section. The cases are then assessed in terms of the oscillatory energy and the mechanisms by which energy enters and departs the system. A discussion of limit cycles is then presented for each of the three cases.

## 2.1 Dynamic Wind Tunnel Testing

Wind tunnel tests techniques [7, 8] are perhaps the oldest and best documented ways of measuring dynamic aerodynamic coefficients. Fundamentally, the techniques are the same as static wind tunnel testing with models being held in a tunnel (in these cases either with a sting or by gravity) as the freestream flow passes over them. Additionally, dynamic testing imposes model motion, or frees the model to respond dynamically on its own, relative to the freestream flow. This model motion is typically oscillatory. Extra forces and moments act upon the model due to these oscillations, in addition to the forces and moments measured in a static test. The dynamic aerodynamic coefficients are recorded as the derivatives of the additional forces and moments introduced by the dynamic motions with respect to the rates at which the model was moving as the data was recorded. These coefficients can be measured directly or extracted from observed model motions, depending on the test technique.

### Free-to-Oscillate

In free-to-oscillate testing, a model is held in a wind tunnel on a sting at a fixed position like traditional static wind tunnel testing. However, the sting also permits oscillatory motion of the model with either bearings or a spring. The model is perturbed from its equilibrium position and the model oscillates freely, either damping down or growing, based on the dynamic stability inherent in the model geometry. Figure 1 shows a typical free-to-oscillate test setup that was used to measure the pitch damping characteristics of the Apollo command module.

Free-to-oscillate testing is very simple to perform and can be modeled very easily for parameter identification. However, it can be difficult to achieve meaningful results. Precise measurement of the capsule position versus time is required, and bearing friction and the presence of a sting can affect the motion of the model, masking the true dynamic behavior. After data is acquired, parameter identification techniques are applied to the model motion history to extract aerodynamic coefficients. A wind tunnel setup is generally a more controlled environment than free flight, potentially producing more repeatable results, with greater control over initial conditions. However, this technique is still limited by not having any direct moment measurements. In that way, the free-to-oscillate technique is similar to ballistic range testing.

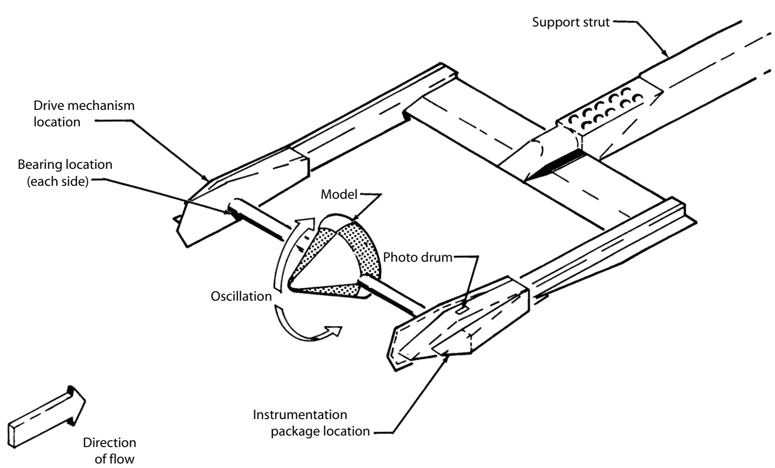
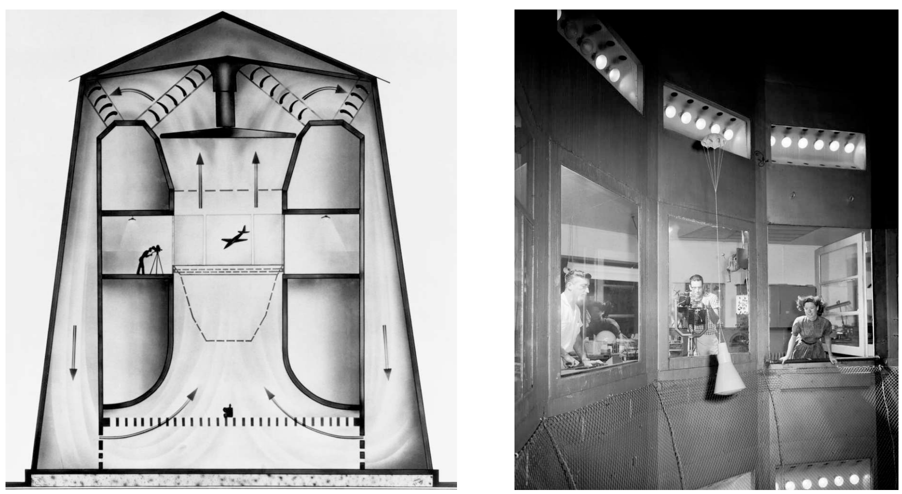



**Figure 1: Free-to-oscillate setup[9]**

### Spin Tunnel

In a vertical spin tunnel the freestream flow is directed vertically upward with the model perpetually “falling” relative to the flow, but not moving with respect to the test section. These facilities are commonly used to measure the terminal-velocity dynamics of blunt bodies at low speeds [10]. Often, the models are dynamically scaled, achieving dynamics very similar to those at flight conditions. Figure 2 shows a cross section of the NASA Langley 20 Foot Spin Tunnel and a test photograph from the Mercury program [11, 12]. In a spin tunnel, the drag acting on the model is balanced by its weight. Angle-of-attack oscillations of the model can cause the drag force to fluctuate which then causes a vertical motion relative to the oncoming flow (and test section). For large changes in drag, control of the freestream velocity may be required to keep the model within the test section. For small oscillations, drag is nearly constant and this test technique achieves constant velocity conditions with the model free to oscillate and heave. Heaving motion is a periodic translation caused by the lift produced as the model oscillates in angle-of-attack. This translational motion results in a flight path angle oscillation about the mean ($\gamma_{mean} = -90°$). It will be shown that this additional freedom of movement results in a change in the effective damping as compared to free-to-oscillate testing where the model is held on a sting.



(a) Cross Section of Tunnel

(b) Mercury Capsule with Drogue Parachute

**Figure 2: NASA Langley 20 Foot Spin Tunnel**

### Forced Oscillation

Forced-oscillation testing is very similar to free-to-oscillate testing, but instead of the model rotating freely, it is forced through prescribed oscillatory motions and the damping can be determined explicitly [7, 8, 13]. Forced oscillation techniques have excellent control over all important test parameters. The model motion is driven by a motor, so it is possible to test a wide range of amplitudes and frequencies without requiring the complicated and sometimes geometry-limiting need to scale the model mass properties. Forced oscillation testing directly measures the forces and moments acting on the vehicle and damping coefficients can be directly calculated from the time histories of those forces and moments. As the motions are prescribed, this is not a method conducive to observing self-limiting oscillations. The data measured with this testing may be more accurate than other methods and can be used to predict amplitude equilibrium oscillations in flight. However, the controlled nature of the testing prevents this technique from being compared in this work.

## 2.2 Ballistic Range Testing

Ballistic range testing is essentially a scaled flight test [1, 2, 14]. Testing is conducted at either indoor ranges where the model motion is tracked photographically or outdoor facilities, where the models are tracked with radar. In an indoor range, models fly freely down an instrumented hallway or chamber, producing flight-like motions from which aerodynamic coefficients are extracted. The models are launched from a gun, held at an initial orientation by a sabot. The gun propels the model to the desired initial velocity and upon exiting the gun barrel, the sabot petals peel away leaving the model to fly down the instrumented portion of the range. The sabot petals are arrested so as not to interfere with downrange measurements. Orthogonal shadowgraphs are taken at multiple stations down the range. Calibrated reference points in the images enable the model position and orientation to be determined at each station. As the spark source illuminating the shadowgraphs is triggered at each station, the time is recorded with a chronograph. Figure 3 shows an example of this setup in the Eglin Air Force Base (AFB) Aeroballistic Research Facility (ARF). The geometry used to identify the capsule position and orientation from shadowgraphs is illustrated for one station in the range.

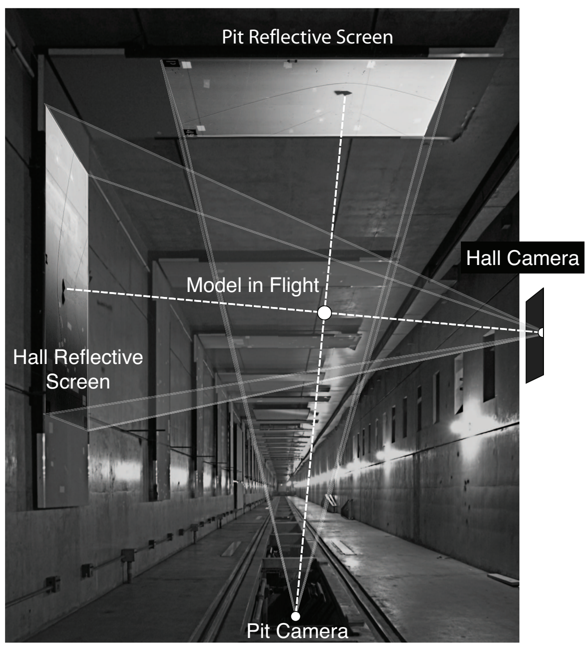



**Figure 3: Looking up-range at the Eglin AFB Aeroballistic Research Facility**

The time, position and orientation of the model at each station is then used to reconstruct the full trajectory. Parameter identification techniques are then used to extract aerodynamic coefficients from the observed trajectories. Multiple shots with different initial oscillation amplitudes and velocities help identify the aerodynamics over a range of angles-of-attack and Mach number. More data helps to more accurately define the functional form of nonlinear aerodynamic coefficients and reduce uncertainties.

For outdoor testing a gun is again used to launch the model. Instrumentation onboard the capsule measures the body rotations and sensed accelerations, while radar tracks the models in flight for independent measurements of position and velocity. The onboard and radar measurements are used to reconstruct the model trajectory. Parameter identification techniques are then used to extract aerodynamics much like what is done for indoor testing. An outdoor range permits the testing of lifting vehicles that would strike the walls of an indoor range. However, as models are not typically recoverable, the costs to instrument each shot can be prohibitive.

While this technique has the overwhelming benefit of achieving flight-like motions, there are several drawbacks and limitations. Chief among them, this test technique does not allow a direct measurement of aerodynamic coefficients. While multiple fits through the data points of many trajectories are done together to find the best fit for the aerodynamic coefficients, the problem is still under-defined. Different functional forms can achieve very similar fits through the data points. This is especially true for the pitch and yaw damping coefficients which tend to be very nonlinear with angle-of-attack and Mach number. Having a good understanding of the relative contributions to capsule motion from drag, lift and damping terms, as well as the effects of mass properties and freestream conditions is very important for interpreting the observed motions for any parameter identification efforts.

## 2.3 Planetary Entry

Ground-based data is gathered for the purpose of predicting flight performance. However, a planetary entry capsule experiences flight conditions that change even more dramatically than those seen by a ballistic range model. While decelerating, the capsule sees increasing, then decreasing dynamic pressure as the freestream density and pressure increase and the velocity decreases along the capsule's descent. Depending on the vehicle mass properties and freestream conditions, the change in flight path angle due to gravity can also affect the capsule dynamics. These more complex effects are beyond the scope of this work, but pointed out to emphasize that first understanding the mechanisms at play in different ground-based test methods is important to interpreting observed oscillatory motions and the extracted dynamic data and predicting how those motions will change in flight.

# 3.0 EQUATIONS OF MOTION

The planar equations of motion for a body flying in a gravity field is the starting point for describing the motions of blunt-body planetary entry and they can be simplified to closely approximate several of the motions seen in ground based testing. The planar motions are described by Equations 1 through 3. Equation 1 sums the forces acting on the body in the direction of motion. Equation 2 describes the change in flight path angle, $\gamma$, due to forces normal to the direction of motion, lift, gravity and centrifugal forces, as the body curves around a planet. Equation 3 sums the inertial, static and dynamic moments acting on the vehicle. These equations are valid for a low lift-to-drag vehicle at small angles-of-attack ($\alpha$ approximately less than 30° for these shapes). Unless otherwise noted, the analysis in this paper assumes that lift and pitching moment vary linearly with angle-of-attack, drag is invariant with angle-of-attack and all aerodynamic coefficients are invariant with velocity/Mach number. The coordinate system for these Equations is shown in Figure 4.

$$\dot{V} = -\frac{\rho V^2 S C_D}{2m} - g \sin\gamma_o \tag{1}$$

$$\dot{\gamma} = \frac{\rho V S C_L}{2m} - \left(\frac{g}{V} - \frac{V}{R}\right)\cos\gamma_o \tag{2}$$

$$\ddot{\theta} = \frac{\rho V^2 S d}{2I}\left(C_{m_q}\frac{\dot{\theta} d}{2V} + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha} d}{2V} + C_{m_\alpha}\alpha\right) \tag{3}$$

Where

$$\theta = \alpha + \gamma \tag{4}$$

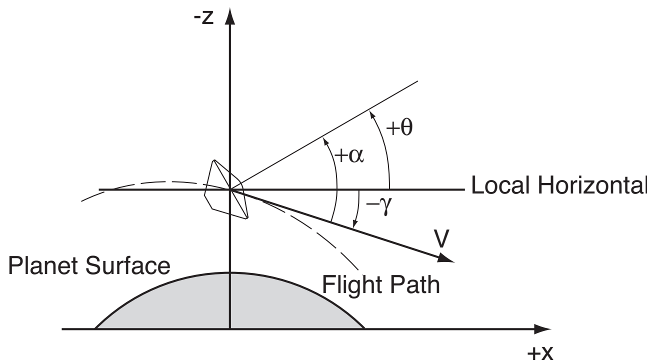



**Figure 4: Coordinate system**

Some simplifying assumptions may be made that permit the moment equation to be cast as a differential equation that can be solved analytically. First, it is assumed that gravity and centrifugal effects are small, meaning the mean flight path angle over a trajectory segment is effectively constant. These assumptions simplify the RHS of Equation 2 to just the contribution due to lift. Flight path angle varying due to lift only is a valid assumption for many ballistic range flights, spin tunnel flight and some free-to-oscillate wind tunnel test conditions. Equation 4, and its first and second derivatives can be used with Equations 1 and 2 to express the moment equation (Equation 3) in terms of angle-of-attack only. First, the LHS becomes

$$\ddot{\alpha} + \left(\frac{\rho V S}{2m}\right)^2 C_D C_{L_\alpha} \alpha + \frac{\rho V S}{2m} C_{L_\alpha} \dot{\alpha} = \frac{\rho V^2 S d}{2I} \left( C_{m_q} \frac{\dot{\theta} d}{2V} + C_{m_{\dot{\alpha}}} \frac{\dot{\alpha} d}{2V} + C_{m_\alpha} \alpha \right) \tag{5}$$

The second term on the LHS of Equation 5 modifies the frequency of oscillation of the system slightly, but is small for the cases presented here and can be neglected. Now, the pitch rate, $\dot{\theta}$, can be expressed in terms of $\dot{\alpha}$ and $\dot{\gamma}$. The first term on the RHS of Equation 5 becomes

$$C_{m_q} \frac{\dot{\theta} d}{2V} = C_{m_q} \left( \frac{\dot{\alpha} d}{2V} + \frac{\rho S d C_L}{4m} \right) \tag{6}$$

The $\dot{\gamma}$ term in Equation 6 is small compared to the $\dot{\alpha}$ term and may be neglected as well. Equation 5 then becomes

$$\ddot{\alpha} - \frac{\rho V S}{2m} \left( -C_{L_\alpha} + \frac{m d^2}{2I} \left( C_{m_q} + C_{m_{\dot{\alpha}}} \right) \right) \dot{\alpha} - \frac{\rho V^2 S d}{2I} C_{m_\alpha} \alpha = 0 \tag{7}$$

This equation can be used to analyze several test techniques, but of the techniques impose further conditions that change the form of its solution. Three cases will be considered. The density and all aerodynamic coefficients are held constant, permitting Equation 7 to be solved analytically for each case. The first case is a free-to-oscillate wind tunnel setup, where the model sees a constant velocity and is free to rotate only. The second case is also a wind tunnel case, where the model sees a constant velocity, but is free to move normal to the freestream velocity vector due to lift in addition to being free to oscillate. For the third case, the model is permitted to decelerate due to drag and is still free to oscillate and heave normal to the freestream velocity vector.

## 4.0 CONSTANT COEFFICIENT ANALYTIC SOLUTIONS

This section will present analytic solutions to the three boundary condition cases just described in Section 3. It will be shown that these different boundary conditions affect the functional form of the solution to Equation 7. For each of these cases a set of common mass properties and initial flow conditions was used. For the constant velocity cases, the conditions remain constant. For the decelerating case, the model decelerates to a final velocity. For each case, plots of oscillation histories for several constant pitch damping values are presented. The time-of-flight for the decelerating case was used as the time interval for all plots. Table 1 lists the mass properties and flow conditions used in the following sections. The initial and final times listed in Table 1 are along a timeline which assumes infinite initial velocity (explained further in Section 4.3). This timeline allows some equation simplification required to obtain an analytical solution for the decelerating case. In all plots below, the timeline is shifted so that the oscillation plots begin at $t_i = 0.0$.

**Table 1: Example Test Parameters**

| Boundary Conditions | | Range/Model Properties | | Aerodynamics | |
|---|---|---|---|---|---|
| $V_i$ | $858\ m/s$ | $m$ | $0.584\ kg$ | $C_{m_\alpha}$ | $-0.09\ rad^{-1}$ |
| $V_f$ | $858\ m/s\ (\dot{V}=0),\ 343\ m/s\ (\dot{V}<0)$ | $I$ | $1.55 \cdot 10^{-4}\ kg \cdot m^2$ | $C_D = C_A$ | $1.58$ |
| $\alpha_o$ | $5°$ | $d$ | $0.07\ m$ | $C_{L_\alpha}$ | $-1.58$ |
| $\dot{\alpha}_o$ | $0\ rad/s$ | $S$ | $0.00385\ m^2$ | $C_{m_q} + C_{m_{\dot{\alpha}}}$ | $+0.15, 0, -0.171, -0.342$ |
| $t_1$ | $0.186\ s\ \ (t_i = 0.0\ s)$ | $\rho$ | $1.20\ kg/m^3$ | $C_N$ | $0.0$ |
| $t_2$ | $0.466\ s\ \ (t_f = 0.28\ s)$ | | | | |

### 4.1 Case 1 : Constant Velocity, Free-to-Oscillate, No-Heave

For this case, Equation 7 must be simplified slightly. As the oscillation center is fixed relative to the freestream flow, there is no change in flight path angle due to the lift generated as the model oscillates, $\dot{\gamma} = 0$. Therefore, the $C_{L_\alpha}$ term, really an expression for $\ddot{\gamma}$ substituted from the time derivative of Equation 2, drops out. This simplifies Equation 7 to

$$\ddot{\alpha} - \frac{\rho V_\infty S d^2}{4I}\left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\dot{\alpha} - \frac{\rho V_\infty^2 S d}{2I} C_{m_\alpha}\alpha = 0 \tag{8}$$

As all coefficients are constant, this equation is a simple harmonic oscillator with damping, having the classic solution

$$\alpha = A e^{\xi_1 t}\cos(\omega t + \delta) \tag{9}$$

where

$$\xi_1 = \frac{\rho V S d^2}{8I}(C_{m_q} + C_{m_{\dot{\alpha}}}) \tag{10}$$

and

$$\omega = \sqrt{-\frac{\rho V^2 S d}{2I} C_{m_\alpha}} \tag{11}$$

Figure 5 shows a plot of Equation 9 for several values of $C_{m_q} + C_{m_{\dot{\alpha}}}$. For this case, understanding the oscillations is very straight forward. Positive values for $C_{m_q} + C_{m_{\dot{\alpha}}}$ cause oscillation amplitude growth, while negative values damp down any oscillations. These test conditions are ideal for isolating and identifying dynamic damping characteristics inherent to the geometry of the vehicle. The dynamic stability is the only term affecting the oscillation amplitude growth.

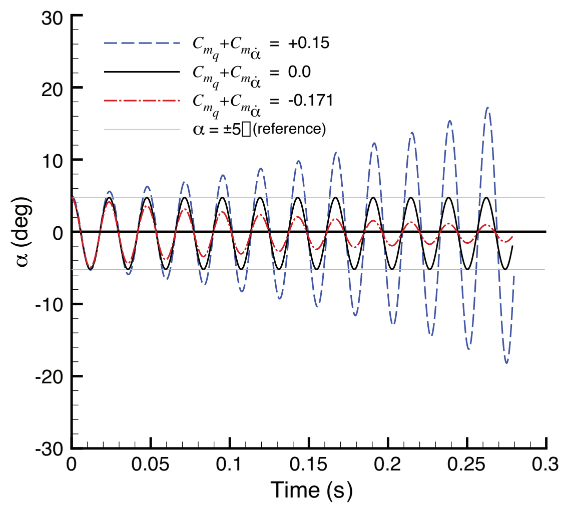



**Figure 5: Oscillation amplitude history for different damping values, constant velocity, no heaving motion.**

## 4.2 Case 2: Constant Velocity, Free-to-Oscillate, Free-to-heave

Here the body is held at constant velocity, but also allowed to move normal to the velocity vector due to the lift generated as the model oscillates rotationally. This extra degree of freedom results in transverse heaving motion which induces changes to the angle-of-attack history. This setup is described correctly by Equation 7. Again we have constant coefficients, so the solution is of the same form,

$$\alpha = A e^{\xi_2 t} \cos(\omega t + \delta) \tag{12}$$

However, $C_{L_\alpha}$ modifies the damping term as

$$\xi_2 = \frac{\rho V S}{4m} \left( -C_{L_\alpha} + \frac{m d^2}{2I} (C_{m_q} + C_{m_{\dot{\alpha}}}) \right) = \xi_1 - \frac{\rho V S}{4m} C_{L_\alpha} \tag{13}$$

Here, we can make one more substitution for blunt bodies at small angles of attack. Recall the relationships between lift and drag, and axial and normal forces

$$\left. \begin{aligned} C_D &= C_A \cos\alpha - C_N \sin\alpha \\ C_L &= C_N \cos\alpha - C_A \sin\alpha \end{aligned} \right\} \tag{14}$$

Linearizing these relations about the capsule trim point puts them in a form which can be further simplify for blunt bodies. For axisymmetric blunt bodies, it is reasonable to assume that the normal force is zero at the trim point and the axial force can be approximated as constant with angle of attack.

$$\left.\begin{array}{r} C_D = C_A - C_{N_\alpha}\alpha^2 + H.O.T. \\ C_L = (C_{N_\alpha} - C_A)\,\alpha + H.O.T. \end{array}\right\} \tag{15}$$

For small to moderate angles of attack ($\alpha \lesssim 30°$), normal forces acting on a blunt body are smaller than axial forces ($C_N \ll C_A$). The $\alpha^2$ term in the $C_D$ expression may be neglected. The lift and drag equations in terms of the normal and axial forces acting on a blunt body can then be simplified. To first order

$$\left.\begin{array}{r} C_D = C_A \\ C_L = (C_{N_\alpha} - C_A)\,\alpha \\ C_{L_\alpha} = (C_{N_\alpha} - C_A) \approx -C_A \end{array}\right\} \tag{16}$$

To simplify the analysis in this work, $C_N$ and therefore $C_{N_\alpha}$ are assumed to be zero. It should be noted that for typical blunt-bodied vehicles, the $C_{N_\alpha}$ term can be a small but noticeable fraction of the lift curve slope. Consider the 70° sphere-cone example case to illustrate the effect of the normal force slope: The supersonic axial force coefficient is 1.58 and in reality, the normal force slope, $C_{N_\alpha}$, is roughly 0.2. The normal force slope decreases the lift-curve-slope magnitude approximately 13% at small angles. Keeping this term would retain extra complexity in some equations developed below, but would not alter the overall conclusions, so for simplicity the normal force slope is neglected. The approximations used here can be considered the "limiting case" for blunt body aerodynamics.
Substituting $-C_A$ for $C_{L_\alpha}$ into Equation 13, the damping term for the constant velocity, free-to-heave case is

$$\xi_2 = \xi_1 + \frac{\rho V S C_A}{4m} \tag{17}$$

Figure 6 shows a plot of Equation 12 for several values of $C_{m_q} + C_{m_{\dot\alpha}}$. As the equations suggest, for no amplitude growth, the dynamic damping coefficient must be negative to balance the undamping due to lift. Retaining the $C_{N_\alpha}$ term would reduce the undamping due to lift and change the exact value requiring a slightly smaller magnitude $C_{m_q} + C_{m_{\dot\alpha}}$ term for zero amplitude growth.

The damping term indicates that allowing the body to move transverse to the oncoming flow results in a more undamped system for blunt bodies. It is important to note that this is fairly specific to blunt bodies only. The lift generated by blunt bodies is almost entirely due to the pointing of the axial force vector. Therefore, to generate an upward lift, the body must be at a negative angle-of-attack. This is opposite to winged aircraft and results in the increased undamping of the system. As a blunt body rotates nose-down and lift causes the body to move up, the upward velocity causes an induced increment to angle of attack. In contrast, a winged vehicle typically pitches up to increase lift. The upward rotation results in an upward motion, inducing a decrease in the angle-of-attack seen by the vehicle. The effects of static aerodynamics on damping will be discussed again in Section 5 when the energy equation is applied to this system.

### 4.3 Case 3: Decelerating, Free-to-Oscillate, Free-to-Heave

Now, the body is allowed to decelerate and is again permitted to oscillate and heave. The heaving degree-of-freedom has an effect similar to that seen in the previous constant velocity case. The no-heave decelerating case will not be shown, as it is not a common test setup. However, a ballistic range test is very closely approximated by Equation 7 if velocity is allowed to vary with time. For this case, velocity decreases due to drag acting on the

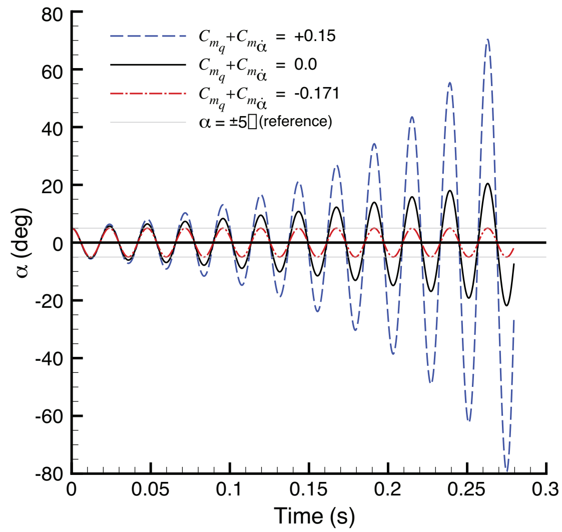



**Figure 6: Oscillation amplitude history for different damping values, constant velocity with heaving motion due to lift.**

vehicle and lift is the only force changing the flight path angle. Again, this is valid for small oscillations. The approximations made in Equation 16 hold. Schoenenberger, Queen and Litton [15] developed the following solution for this decelerating case. The results presented here are explained in more detail in that work. For a constant drag coefficient (invariant with small oscillations), the deceleration of a body can be expressed as

$$V = \frac{2m}{\rho S C_A t} \tag{18}$$

Substituting this expression into Equation 7 with some rearranging yields

$$t^2\ddot{\alpha} - \left(1 + \frac{md^2}{2I}\frac{\left(C_{m_q} + C_{m_{\dot\alpha}}\right)}{C_A}\right)t\dot{\alpha} - \frac{2m^2 d C_{m_\alpha}}{\rho S I C_A^2}\alpha = 0 \tag{19}$$

Note, in Equation 18, velocity is infinite at time, $t = 0$. Solving Equation 19 must be done on a timeline that starts at infinite velocity. This is handled by solving Equation 18 for an initial time, $t_i > 0$, that corresponds to the desired initial velocity.

$$t_i = \frac{2m}{\rho V_i S C_A} \tag{20}$$

While awkward, the simplified version of the deceleration equation yields a differential equation with an analytic solution. Equation 19 is the Euler-Cauchy equation [16] and has the following solution

$$\alpha = At^{\mu}\cos(\nu \ln t + \delta) \tag{21}$$

Where $A$ is a constant determined by the oscillation amplitude at the boundary conditions and the constant, $\delta$, is a phase lag, also determined from the boundary conditions. The coefficient, $\nu$, is

$$\nu = \sqrt{\mu^2 - \frac{2mdC_{m_\alpha}}{\rho S I C_A^2}} \approx \sqrt{-\frac{2m^2 d C_{m_\alpha}}{\rho S I C_A^2}} \tag{22}$$

Oscillation amplitude grows with time, raised the the power, $\mu$. The coefficient, $\mu$, is

$$\mu = \frac{md^2\left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)}{4IC_A} + 1 \tag{23}$$

As with the constant velocity version, for the aerodynamics of typical blunt bodies, the damping moments in Equation 19 are very small compared to the static pitching moments. Therefore the coefficient, $\mu$, may be neglected in Equation 22.

For a body with a pitch damping coefficient of zero ($C_{m_q}+C_{m_{\dot{\alpha}}} = 0$), $\mu = 1$ and oscillation amplitude will grow in direct proportion to time ($\alpha_o \propto t$) while the velocity decreases as $1/t$. As with the constant velocity case, the change in flight path angle due to the lift curve slope, $-C_{L_\alpha}$, results in an oscillation amplitude growth even without a contribution from the dynamic stability coefficient. However, the functional form of the oscillation growth has changed.

Figure 7 shows a plot of Equation 21 for several values of $C_{m_q} + C_{m_{\dot{\alpha}}}$. As the equations suggest, for no amplitude growth, the dynamic damping must be negative to counteract the undamping due to lift and natural growth due to decreasing dynamic pressure. The value of $C_{m_q} + C_{m_{\dot{\alpha}}}$ required for no amplitude growth is exactly twice that for the constant velocity, free-to-heave case.

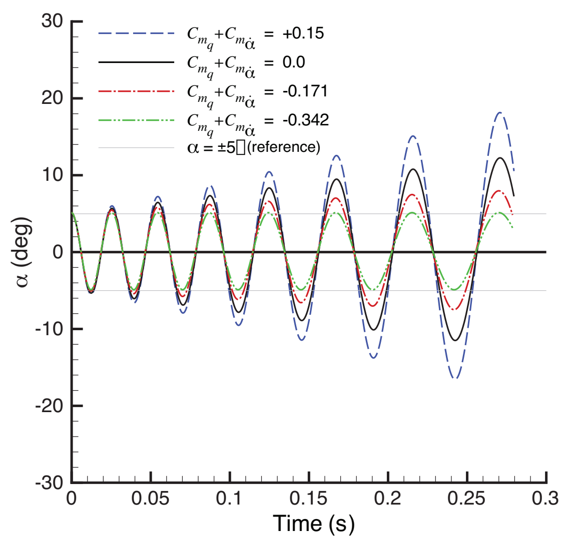



**Figure 7: Oscillation amplitude history for different damping values, decelerating with heaving motion due to lift.**

## 5.0 ENERGY EQUATION ANALYSIS

Here we will look at the oscillatory energy of the three different cases representing different wind tunnel and ballistic range test conditions. This use of the energy equation will highlight the mechanisms that drive the oscillation growth or decay for different flight or test conditions. The time variation of energy will be derived for each case. The moment equation will help simplify the energy equation and show the system to be either conservative or not depending on the imposed flight conditions. The evaluation of each system will show how the dynamic stability and static aerodynamics contribute to the oscillatory energy. Later, an approach to find the effective damping for blunt bodies with nonlinear dynamic stability characteristics will be applied with this energy-equation analysis to predict equilibrium oscillations and constant energy oscillatory behavior.

In the inertial frame, the equation describing the oscillation energy of a free flying body is the sum of the kinetic and potential oscillatory energy:

$$K = \frac{1}{2}I\dot{\theta}^2 - \frac{1}{4}\rho V^2 S d C_{m_\alpha}\alpha^2 \tag{24}$$

The rate of change of this energy is then

$$\frac{dK}{dt} = I\ddot{\theta}\dot{\theta} - \frac{1}{2}\rho V^2 S d C_{m_\alpha}\alpha\dot{\alpha} - \frac{1}{2}\rho V S d C_{m_\alpha}\alpha^2\dot{V} \tag{25}$$

Now, the moment equation with dynamic damping is

$$I\ddot{\theta} - \frac{1}{2}\rho V^2 S d\left((C_{m_q} + C_{m_{\dot{\alpha}}})\frac{\dot{\alpha}d}{2V} + C_{m_\alpha}\alpha\right) = 0 \tag{26}$$

Multiplying the moment equation by $\dot{\theta}$ and expanding yields

$$I\ddot{\theta}\dot{\theta} = \frac{1}{2}\rho V^2 S d\left((C_{m_q} + C_{m_{\dot{\alpha}}})\frac{(\dot{\alpha}^2 + \dot{\alpha}\dot{\gamma})d}{2V} + C_{m_\alpha}\alpha(\dot{\alpha} + \dot{\gamma})\right) \tag{27}$$

Substituting Equation 27 into Equation 25 yields

$$\frac{dK}{dt} = \frac{1}{2}\rho V^2 S d\left((C_{m_q} + C_{m_{\dot{\alpha}}})\frac{(\dot{\alpha}^2 + \dot{\alpha}\dot{\gamma})d}{2V}\right) + \frac{1}{2}\rho V^2 S d C_{m_\alpha}\alpha\dot{\gamma} - \frac{1}{2}\rho V S d C_{m_\alpha}\alpha^2\dot{V} \tag{28}$$

From the planar equations, the change in flight path angle can be linearized

$$\dot{\gamma} = \frac{\rho V S C_L}{2m} = \frac{\rho V S C_{L_\alpha}}{2m}\alpha \approx -\frac{\rho V S C_A}{2m}\alpha \tag{29}$$

and for a decelerating body

$$\dot{V} = -\frac{\rho V^2 S C_A}{2m} \approx \dot{\gamma}\frac{V}{\alpha} \tag{30}$$

For a body moving at constant velocity, the change in velocity is of course zero

$$\dot{V} = 0 \tag{31}$$

Equations 28-31 provide the tools needed to assess the energy rates for the three blunt body oscillation cases.

**Constant Velocity, No-Heave**

For a constant velocity case that is not allowed to heave we have

$$\dot{V} = 0, \quad \dot{\gamma} = 0 \tag{32}$$

Equation 28 simplifies to

$$\frac{dK}{dt} = \frac{1}{4}\rho V S d^2(C_{m_q} + C_{m_{\dot{\alpha}}})\dot{\alpha}^2 \tag{33}$$

Energy enters the system only by the dynamic damping term. Integrating the change in energy over an entire cycle for this case yields

$$K_f = \frac{1}{4}\rho V S d^2 \int_0^{\frac{2\pi}{\omega}} \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right) \dot{\alpha}^2 dt + K_i \tag{34}$$

For $C_{m_q} + C_{m_{\dot{\alpha}}} = 0$, no energy is entering the system and $K_f$ in Equation 34 is trivially equal to the initial energy of the system. Later, nonlinear pitch damping curves will be evaluated using this equation to find amplitudes of constant oscillatory energy.

### Constant Velocity, Free-to-Heave, No Dynamic Damping

This case approximates terminal velocity flight, or a vertical wind tunnel test, where drag and gravity's pull are in equilibrium, resulting in constant freestream velocity, yet still permitting motions normal to the oncoming flow. In practice, there may be coupling of the oscillatory motion and descent rate as the small angle approximations may be violated. This coupling is neglected here.

By allowing the model to change flight path angle due to lift (heave) at constant velocity, the problem becomes slightly more complicated than the previous case. First, consider this case with no dynamic damping. Equation 28 becomes

$$\frac{dK}{dt} = \frac{1}{2}\rho V^2 S d C_{m_\alpha} \alpha \dot{\gamma} = \frac{1}{2}\rho V^2 S d C_{m_\alpha} \alpha^2 \frac{\rho S V}{2m} C_{L_\alpha} \tag{35}$$

Even with no dynamic damping, rotational energy enters the system. To keep the model from decelerating, a sting or some other force (like gravity) must balance the drag force (approximately equal to the axial force for small oscillations). When the model oscillates, the lift generated by the body takes energy from the oncoming flow and converts it to oscillatory energy. Note that for blunt bodies, the lift curve slope is negative, so the change in energy is positive For a winged vehicle, the lift curve slope is typically positive, which results in the removal of oscillatory energy from the system. The static lift characteristics of blunt bodies, driven by the pointing of the axial force vector, results in an undamped bias to their dynamic stability. The steeper the lift curve slope, the greater the bias.

The amount of energy that goes into the system is proportional to the dynamic pressure of the flow and the ratio of the approaching mass flux to the model mass. So, a model with a high density will see less transverse acceleration and therefore convert less of the energy in the freestream flow to oscillatory energy than a lower density body. The pitching moment slope and axial force coefficient also determine how efficiently the geometry of a particular body converts the freestream flow energy into oscillations.

### Constant Velocity, Free-to-Heave, with Dynamic Damping

Now, the case with dynamic damping is considered. In this case, the $\dot{V}$ term in equation 28 is the only one that drops out immediately. Dropping the $\dot{V}$ term and rearranging we have

$$\frac{dK}{dt} = \frac{1}{2}\rho V^2 S d \left(C_{m_\alpha}\alpha + \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\frac{\dot{\alpha} d}{2V}\right)\dot{\gamma} + \frac{1}{2}\rho V^2 S d \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\frac{\dot{\alpha}^2 d}{2V} \tag{36}$$

Substituting the expression for $\dot{\gamma}$, Equation 2, into Equation 36 and again rearranging yields

$$\frac{dK}{dt} = \frac{\rho V S I}{2m}\left(\frac{\rho V^2 S d}{2I} C_{m_\alpha} C_{L_\alpha} \alpha^2 + \frac{m d^2}{2I}\left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\dot{\alpha}^2\right) + \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\frac{1}{2}\rho V^2 S d \frac{\rho S d}{4m} C_{L_\alpha} \alpha \dot{\alpha} \tag{37}$$

For oscillations of constant amplitude at constant velocity, where the damping is not a large component of the total pitching moment, the angle-of-attack history can be represented as a sinusoid.

$$\alpha = A\cos(\omega t) \tag{38}$$

$$\dot{\alpha} = -A\omega\sin(\omega t) \tag{39}$$

Here, the frequency of oscillation, $\omega$, from the static stability term in the moment equation, given in Equation 11 is used. Substituting Equation 11 into Equation 37 yields

$$\frac{dK}{dt} = \frac{\rho VSI}{2m}\left(-C_{L_\alpha}\omega^2\alpha^2 + \frac{md^2}{2I}\left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\dot{\alpha}^2\right) + \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\frac{1}{2}\rho V^2 Sd\frac{\rho Sd}{4m}C_{L_\alpha}\alpha\dot{\alpha} \tag{40}$$

Now an expression for the total rotational energy entering the system (the integral of Equation 40) may be set to zero to find the damping required for a limit cycle. The energy rate must be integrated over a complete cycle. With that constraint, and Equations 38 and 39 for $\alpha$ and $\dot{\alpha}$, the following relation holds,

$$\int_0^{\frac{2\pi}{\omega}} \omega^2\alpha^2 dt = \int_0^{\frac{2\pi}{\omega}} \dot{\alpha}^2 dt \tag{41}$$

After substituting Equation 41 into Equation 40, the integral is taken to obtain the total oscillatory energy added to a constant velocity, free-to-heave system with damping. For no oscillation amplitude growth for this constant velocity case, the net energy added must be zero.

$$K_f = \frac{\rho VSI}{2m}\int_0^{\frac{2\pi}{\omega}}\left(-C_{L_\alpha} + \frac{md^2}{2I}\left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\right)\dot{\alpha}^2 dt + K_i = 0 \tag{42}$$

Note that the integral of the $\alpha\dot{\alpha}$ term in Equation 40 disappears when integrated over a complete cycle. Recalling Equation 7, this is exactly the integral of the moment equation, expressed in terms of angle-of-attack only, multiplied by the rate of change of angle-of-attack, $\dot{\alpha}$. Thus, it has been shown that for a constant velocity system, the dynamic stability of the blunt body must be negative to balance the energy added due to heaving motions, induced ultimately by the force required to maintain the constant velocity. The particular static aerodynamics of the vehicle determine how effectively the drag force is converted to oscillations. In that sense the amplitude of a limit cycle is driven by both the dynamic stability characteristics and the static aerodynamics.

**Decelerating, Free-to-Heave**

The decelerating case is the last case for consideration. Substituting Equation 18 into 2 and 1 yields

$$\dot{\gamma} \approx -\frac{\alpha}{t} \tag{43}$$

$$\dot{V} = -\frac{V}{t} \tag{44}$$

These expressions can then be substituted to simplify Equation 28

$$\frac{dK}{dt} = \frac{md^2}{2C_A t}\left((C_{m_q} + C_{m_{\dot{\alpha}}})(\dot{\alpha}^2 - \frac{\dot{\alpha}\alpha}{t})\right) - \frac{1}{2}\rho V^2 SdC_{m_\alpha}\frac{\alpha^2}{t} + \frac{1}{2}\rho V^2 SdC_{m_\alpha}\frac{\alpha^2}{t} \tag{45}$$

The last two terms obviously cancel leaving only a dynamic damping term to add or remove energy from the oscillatory system. The $\frac{\dot{\alpha}\alpha}{t}$ term in Equation 45 modifies the rate at which dynamic damping adds or removes oscillatory energy in the system. When integrated over a full cycle this term is small and may be neglected. The trivial case, $C_{m_q} + C_{m_{\dot{\alpha}}} = 0$, results in a conservative system, $\frac{dK}{dt} = 0$, even though the analytical solution shows that the oscillation amplitude grows linearly with time (Equation 21). The energy entering the system due to the body lifting is negated by the decrease in dynamic pressure as the body decelerates.

This is the important finding for the decelerating case; even with out dynamic instability, there is an oscillation growth. For the blunt-body limiting case, where the lift curve slope is dominated by the axial force, the oscillation growth is conservative. Referring back to Section 4.2, if the $C_{N_\alpha}$ term is retained in the lift curve slope, the oscillation energy decreases slightly even though the oscillation amplitude growth is still positive

$$\frac{dK}{dt} = \frac{1}{2}\rho V^2 S d C_{m_\alpha} \frac{\alpha^2}{t} \frac{C_{N_\alpha}}{C_A} \tag{46}$$

The $C_{N_\alpha}$ term decreases the effectiveness of a blunt body in converting point-mass kinetic energy into oscillatory energy. This decrement is on the order of 10% for common blunt bodies.

## 6.0 PITCH DAMPING CHARACTERISTICS OF BLUNT BODIES

To apply the findings in Sections 4 and 5 to more realistic cases, it is important to discuss the dynamic damping coefficients typical of blunt bodies. In the supersonic regime, the pitch and yaw damping of blunt bodies can be very nonlinear. Some typical pitch damping curves are presented here and the mean value theorem is applied to average the effect of the dynamic damping coefficients over a full oscillation cycle.

### 6.1 Typical Pitch Damping Characteristics

Figure 8 shows several pitch damping curves extracted from ballistic range tests of the Mars Exploration Rover (MER) entry capsule [3]. The $C_{m_q} + C_{m_{\dot{\alpha}}}$ values are positive at small angles of attack, crossing zero and becoming negative at large angles. The capsule is negatively damped at small angles which tends to increase oscillation amplitudes until the undamped region is balanced by the positive damping at large angles, thus reaching equilibrium in oscillation amplitude. Data for several Mach numbers are shown. As the capsule decelerates the undamped region becomes more unstable and extends to greater angles-of-attack. The features in Figure 8 are typical of many blunt bodies. The analytic solutions developed earlier can not be applied directly to assess or compare the dynamic characteristics of blunt bodies with such nonlinear pitch damping curves. The mean value theorem will now be used to calculate constant-value equivalents to these curves as a function of oscillation amplitude. The dynamic damping curves will then be cast in a form suitable for interpretation using the analytic solutions.

### 6.2 Mean Value Theorem Averaging

For the nonconservative systems described in Section 5, the rate at which energy is added or removed is a function of the dynamic stability, multiplied by the square of the angular rate. To evaluate the nonlinear damping properties of blunt bodies, it is useful to integrate the dynamic damping effects over an entire cycle. The mean value theorem allows a very nonlinear curve to be replaced with a constant value for oscillations of constant amplitude and can be applied for a few oscillations where the amplitude is not growing or decaying significantly.

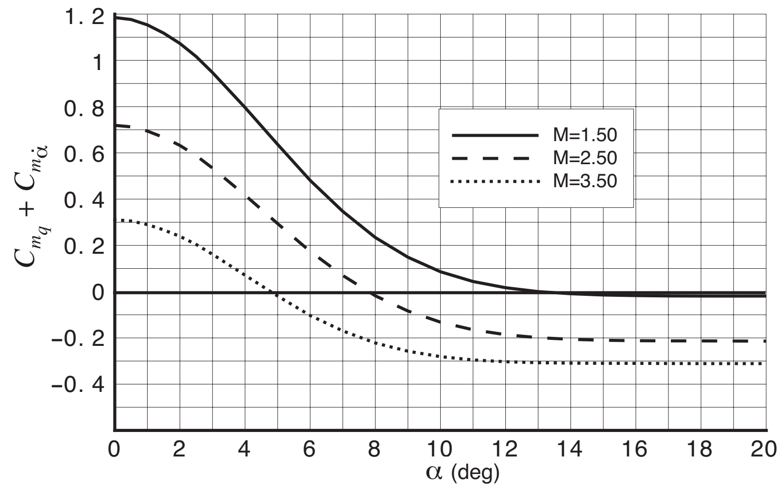



**Figure 8: Example pitch damping data from Mars Exploration Rover ballistic range data,** $x_{cg}/D = 0.30$.

This averaging can find the amplitude where dynamic stability is effectively zero or can find the damping required to balance the lift-curve slope and deceleration effects.

Redd et al [17] showed that system damping characterized by a function of instantaneous angle-of-attack can also be represented as a mean value that varies as a function of oscillation amplitude. Redd made the reasonable assumption that the energy entering or leaving the system per cycle is the same whether damping is a function of instantaneous angle-of-attack or oscillation amplitude. This enabled the use of the mean value theorem to average any nonlinear damping function to obtain an effective damping coefficient for a given oscillaton amplitude. This effective damping can be substituted for the nonlinear curve and the system will act the same.

$$\int_0^T f(\alpha)\dot{\alpha}^2 dt = f(\alpha_o)\int_0^T \dot{\alpha}^2 dt \tag{47}$$

$$f(\alpha_o) = \frac{\int_0^T f(\alpha)\dot{\alpha}^2 dt}{\int_0^T \dot{\alpha}^2 dt} \tag{48}$$

This approach can be applied to any of the energy rate equations presented in Section 5. Integrating the power over a full cycle can be used to find the constant-amplitude equilibrium point or find an amplitude where no net energy is entering the system (not necessarily the same, depending on test conditions) for a blunt body with nonlinear damping.

Recall the constant velocity, no-heave and free-to-heave energy rate equations (Equations 33 and 42). By casting these power equations in the form of Equation 48, the effective damping as a function of oscillation amplitude may be determined.

For the constant velocity, no-heave case the effective damping is

$$\overline{C_{m_q}} = \frac{\int_0^{\frac{2\pi}{\omega}} \left(C_{m_q} + C_{m_{\dot\alpha}}\right)\dot{\alpha}^2 dt}{\int_0^{\frac{2\pi}{\omega}} \dot{\alpha}^2 dt} \tag{49}$$

Likewise the constant velocity, free-to-heave effective damping is

$$\overline{-C_{L_\alpha} + C_{m_q}} = \frac{\int_0^{\frac{2\pi}{\omega}} \left(-\frac{2I}{md^2} C_{L_\alpha} + \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\right) \dot{\alpha}^2 dt}{\int_0^{\frac{2\pi}{\omega}} \dot{\alpha}^2 dt} \tag{50}$$

Where $\overline{C_{m_q}}$ and $\overline{-C_{L_\alpha} + C_{m_q}}$ are shorthand terms used to represent the mean effective damping determined by evaluating the RHS of equations 49 and 50 for a particular oscillation amplitude, $\alpha_o$. To analytically solve these expressions, the lift curve slope and nonlinear pitch damping must be modeled as a function of angle of attack (itself a periodic function of time) that yields an integral in the numerator that may be evaluated analytically. Otherwise, the expression must be evaluated numerically. For small amplitude growth over a single cycle, this relation can be used to assess the effective damping away from the equilibrium oscillation amplitude. The oscillation amplitude for which Equation 50 is equal to zero is where the dynamic damping balances the $C_{L_\alpha}$ contribution, satisfying Equation 42.

For the example used throughout this paper, the lift curve slope is assumed constant. In this case, Equation 50 can be simplified even further

$$\overline{-C_{L_\alpha} + C_{m_q}} = -\frac{2I}{md^2} C_{L_\alpha} + \overline{C_{m_q}} \tag{51}$$

For a given oscillation amplitude, the effective damping for a constant velocity, free-to-heave system is identical to the damping of the constant velocity, no-heave system, offset by a constant factor. For a blunt body with nonlinear dynamics like those described in Section 6.1, and all other conditions being equal, a blunt body that is free-to-heave will reach oscillatory equilibrium at a larger amplitude than for a system that is constrained against translation from lift.

## 7.0 LIMIT CYCLE DISCUSSION

Discussions about oscillation amplitudes and energy to this point have been careful to avoid the term "limit cycle". This is because for some of the cases presented, a constant amplitude oscillation does not correspond to a constant energy system. Conversely, some constant-energy systems grow in amplitude. Recalling the effective damping equations in the previous section, both were derived from the energy equation. Evaluating Equations 49 or 50 to find where $\overline{C_{m_q}}$ or $\overline{-C_{L_\alpha} + C_{m_q}}$ is zero, finds the point where oscillation amplitude is constant and there is no net energy entering or leaving the system. However, this does not hold for the decelerating case. Applying mean value theorem to the decelerating, free-to-heave energy equation (Equation 45) yields

$$\overline{C_{m_{q,decel}}} = \frac{\int_{t_1=e^{-\frac{\delta}{\nu}}}^{t_2=e^{\frac{2\pi-\delta}{\nu}}} \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right) \left(\frac{\dot{\alpha}^2}{t} - \frac{\dot{\alpha}\alpha}{t^2}\right) dt}{\int_{t_1=e^{-\frac{\delta}{\nu}}}^{t_2=e^{\frac{2\pi-\delta}{\nu}}} \left(\frac{\dot{\alpha}^2}{t} - \frac{\dot{\alpha}\alpha}{t^2}\right) dt} \tag{52}$$

Note that the limits of integration are those for a full cycle of the analytic solution of the Euler-Cauchy equation, derived for the decelerating case (Equation 21). Evaluating this expression to find an amplitude where $\overline{C_{m_{q,decel}}}$ is zero does find the amplitude where no net energy is entering the oscillatory system. However, as was shown in the analytical solution for the decelerating case, $C_{m_q} = 0$ does not correspond to a constant amplitude oscillation. The dynamic pressure is dropping, decreasing the "spring stiffness" of the system and oscillation amplitude grows linearly. For this case, both oscillation amplitude and velocity are changing with time for a constant energy system. Nonlinear pitch damping curves typically vary significantly with both amplitude and Mach number, so it is difficult to use the mean value theorem to find an average damping coefficient that resulted in no net energy over more than a few oscillation cycles.

The exponent, $\mu$, in Equation 21, can be used to predict the constant oscillation amplitude for decelerating, free-to-heave system with nonlinear dynamic damping. The mean value theorem allows an average value to replace a nonlinear function for a given oscillation amplitude. The effective damping that satisfies $\mu = 0$ should result in a constant oscillation amplitude and is the damping that results in the net energy loss required to do so. This is analogous to the constant offset difference between the free-to-heave and no-heave constant velocity cases. Setting $\mu$ to zero in Equation 23 and rearranging yields

$$C_{m_q} + C_{m_{\dot{\alpha}}}\Big|_{\alpha_0 = constant} = -\frac{4IC_A}{md^2} \approx \frac{4I}{md^2} C_{L_\alpha} \tag{53}$$

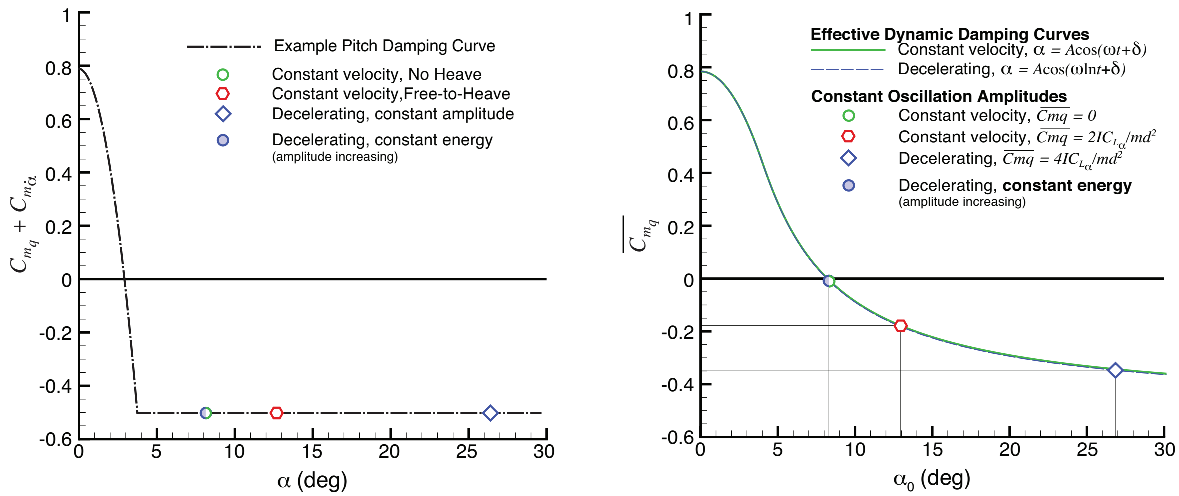
This is twice the factor by which the two constant velocity cases differed. To be at an equilibrum oscillation amplitude, a decelerating blunt body that is free to heave must have positive dynamic damping ($\overline{C_{m_q}} < 0$), twice that needed for the constant velocity, free-to-heave case. This gets at the role of dynamic damping and its relation to oscillation amplitude and energy under different test conditions. For all of the analysis presented here, the oscillatory energy is the only energy of the system being considered. In reality, the point-mass kinetic energy history and whether that energy is coupled to the oscillatory energy is critical to understanding the oscillatory behavior of systems. The point-mass kinetic energy of a system may or may not be constant and can be much greater in magnitude than the oscillatory energy. The conditions of the constant velocity, free-to-heave case closely approximate a blunt body falling at terminal velocity. For that case, the loss of potential energy is essentially balanced by the work done on the body by aerodynamic drag. The small lift force generated as the model decelerates “bleeds” a small amount of the kinetic energy into the oscillatory system. This amount is so small, so as not to affect the terminal velocity significantly, but large enough to have a very noticeable effect on the the oscillatory behavior of the system. With no dynamic damping or nonlinear aerodynamics countering the energy being added to the system from lift, a blunt body at terminal velocity will eventually flip over. For a decelerating case, the point-mass kinetic energy of the system is decreasing as aerodynamic forces decelerate the body. This deceleration reduces dynamic pressure and causes oscillations to grow in amplitude and drop in frequency. Even neglecting the energy “cross-talk” due to lift, the point-mass change in kinetic energy of the system affects the oscillatory behavior. A decelerating case must have positive dynamic damping to keep oscillations from growing in amplitude.

## 7.1 Example Case

### Pitch Damping Model

To illustrate the application of the mean value theorem in predicting oscillation equilibrium, a nonlinear pitch damping curve will be evaluated. This curve replaces the constant values in the original example first described in Section 4. For these cases the time-of-flight is extended to one second to show the convergence towards oscillation equilibrium. All other conditions described in Table 1 are unchanged. Figure 9a shows the example nonlinear pitch damping curve used for this analysis. The curve is parabolic at small angles, switching to a constant, negative value at all large angles of attack. While made-up, it is representative of pitch damping curves seen in blunt body dynamic testing [2]. Figure 9b shows the effective damping for a range of oscillation amplitudes determined by the application of Equations 49 and 52 to the nonlinear damping curve, $C_{m_q}(\alpha)$. Note that integrating over complete constant velocity and decelerating cycles produce almost identical effective damping curves. The points on the plot are the predicted equilibrium oscillation amplitudes for the three cases.

Also noted is the amplitude at which the decelerating case sees no net energy addition. As shown in Section 4, the oscillation amplitude for this case is still increasing. Figure 9b suggests that the amplitude growth-rate will decrease as the capsule passes through this constant-energy amplitude, eventually settling at constant amplitude oscillations. The effective damping coefficient, $\overline{C_{m_q,}}$ becomes negative as the body spends more time in the dynamically damped region of the $C_{m_q} + C_{m_{\dot\alpha}}$ curve.




(a) Damping as function of instantaneous angle of attack (b) Damping as function of amplitude

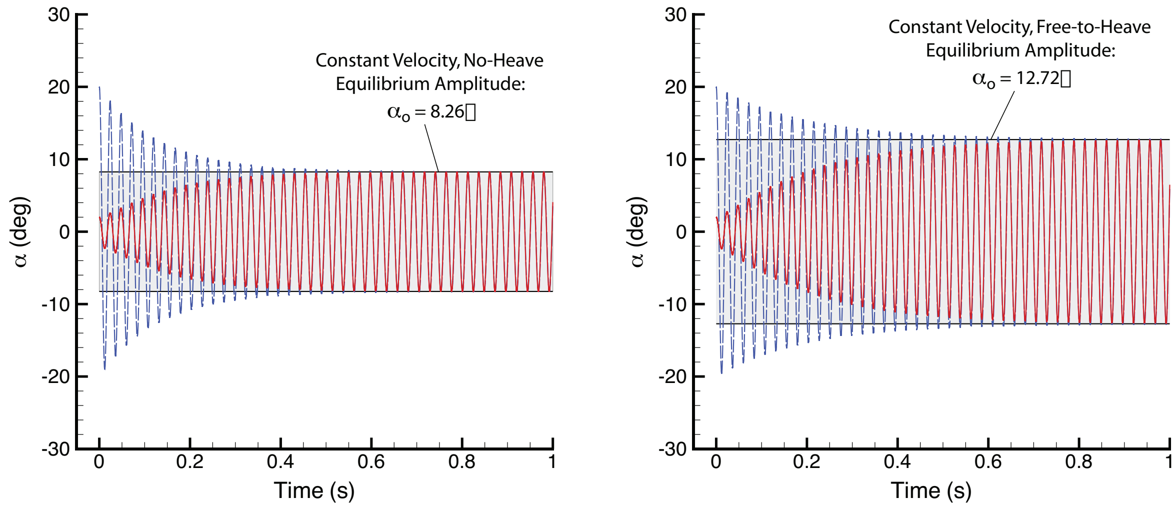
**Figure 9: Pitch damping curve and integrated effective damping, noting constant amplitude and energy points for three cases.**

## Simulations

To demonstrate the effect of the example nonlinear damping, some simulations of the three test cases are presented. For the two constant velocity cases, the model is placed at initial angles above and below the equilibrium amplitudes predicted in Figure 9. For the decelerating case, initial angles above, below and at the predicted equilibrium oscillation amplitude are used.

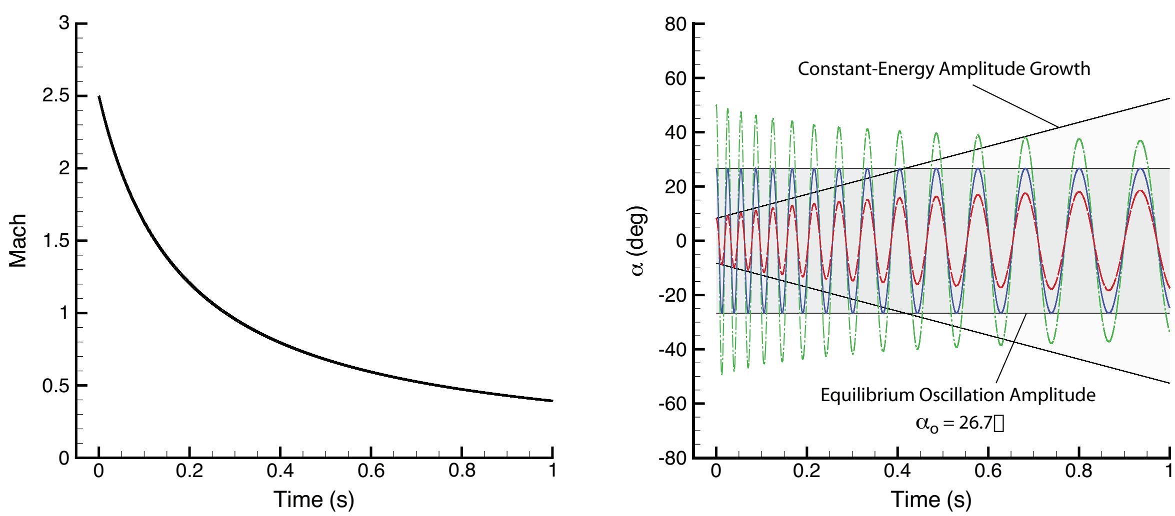
Figure 10 shows simulations of the constant velocity cases at Mach = 2.5. Initial angles-of-attack are 20° and 2° degrees for both. All simulations reach the predicted oscillation amplitude for the particular boundary conditions. Note that the case where lift contributes to the oscillatory energy takes longer to reach equilibrium. Equation 51 shows that the lifting term effectively shifts the dynamic stability curve by a positive bias. This shift has two effects. First the positive region of the pitch damping curve is bigger, meaning the capsule will grow to a larger amplitude before reaching equilibrium (as shown). More energy enters the system at small angles. Second, the negative portion of the curve is shifted up, becoming less effective at damping oscillations toward equilibrium. For this particular example, if the lift curve slope were steep enough, the effective damping could be completely positive resulting in oscillation divergence.

Figure 11 shows simulations of the decelerating system. Note that the Mach number of the model, as it decelerates, drops from an initial value of Mach=2.5 down to approximately Mach=0.40. As was seen in the analytical solutions in Section 4, the frequency of oscillation decreases as the model slows down, but the amplitude converges towards an equilibrium amplitude consistent with the prediction in Figure 9. The shift in the effective damping due to lifting effects occurs in this case as in the constant velocity simulation. In addition, the decrease in dynamic pressure further reduces the damping of the system. This is supported by Equation 45




(a) Constant velocity, no translation due to lift (b) Constant velocity, free-to-heave

**Figure 10: Constant velocity oscillations approaching predicted equilibrium oscillations, $\alpha_i = 2^\circ, 20^\circ$.**




(a) Mach history (b) Decelerating, free-to-heave

**Figure 11: Simulations of decelerating, free-to-heave system with predicted constant energy and equilibrium amplitude limits, $\alpha_i = 8.3,\ 26.7^\circ$ and $50^\circ$. Mach vs. time is shown for reference.**

which shows the energy rate for this system to be decreasing with time. As the conditions shown here are representative of real ballistic range flight conditions, mass properties and aerodynamics, it is clear that direct observation of the steady-state amplitude can be difficult to achieve.

The low initial angle-of-attack case was started at the predicted constant energy amplitude (see Figure 9). This is the angle of attack where pitch damping does not add or subtract energy from the system ($dK/dt = 0$) and oscillations grow due to the drop in dynamic pressure only. The predicted amplitude growth for $\overline{C_{m_q}} = 0$ is plotted in Figure 11b. For approximately one cycle the simulation follows the constant energy prediction fairly closely. As the amplitude grows the negative portion of the nonlinear pitch damping curve contributes more to the overall damping and the oscillations grow less rapidly.

As dynamic pressure drops, the damping becomes less and less effective. Therefore, large displacements from the equilibrium amplitude take a long time to damp down or grow up to reach constant amplitude oscillations. To verify the predicted equilibrium oscillations, a case starting at the predicted value was run. As expected, the capsule remained at the initial amplitude. It is important to keep in mind that the oscillatory energy of this constant-amplitude system is continually decreasing with the drop in dynamic pressure.

## 8.0 CONCLUSIONS

This work has shown analytic solutions for three blunt-body oscillatory systems that are representative of test techniques frequently used to measure dynamic stability. Using the analytic solutions and the energy equation, the effects of the boundary conditions for each setup on the oscillatory behavior were determined. This should help the test engineer interpret experimental results of a particular vehicle by better understanding the effects imposed by the test technique and differentiating between the different contributors to oscillatory motions. One key feature of blunt body aerodynamics is that the lift curve slope is typically negative. This results in a natural undamping of the system when the body is free to translate due to lift. This is in contrast to a winged vehicle with a positive lift curve slope, which sees a natural positive damping for any disturbance from trim.

When a model is free to decelerate, the proper definition of “limit cycle” becomes somewhat murky. A decelerating system with no dynamic damping will grow in amplitude, yet the oscillatory energy of the system is constant. A damped decelerating system that reaches an equilibrium amplitude is losing oscillatory energy as the frequency of oscillation is decreasing due to the drop in dynamic pressure. This paper has been careful to refer to either an oscillation equilibrium amplitude, or constant energy oscillation growth. A limit cycle seems to imply that an equilibrium oscillation amplitude is reached, but should also correspond to a balance of the inputs to the oscillatory energy. For the decelerating system these two conditions are not coincident. When interpreting test results it is good to look for regions where nonlinear dynamic damping is in equilibrium, while keeping in mind the test conditions and how they modify the dynamic behavior. Dynamics can be affected by translation from lift as well as the change in dynamic pressure from deceleration.

Several effects were not addressed in this work. Density varies greatly along a planetary entry trajectory. This will modify the dynamic pressure differently than the constant density analysis shown here. Early along an entry profile, dynamic pressure is increasing due to density and freestream pressure rise, even though the capsule is decelerating. These effects should be considered in future work. Also, pitch damping and lift can vary significantly with Mach number. This complicates the analysis presented here. In practice, a large number of ballistic range or wind tunnel test conditions are required to fully determine the nonlinear pitch damping characteristics of blunt bodies.

There is currently no formal methodology for building ballistic range test matrices. There has historically been a lack of control of initial conditions, driven in part by sabot separation dynamics. In general a test engineer strives to obtain data with the test article oscillating about a number of different amplitudes and at different velocities to anchor multiple-fit 6-DoF simulations, solving for the aerodynamic coefficients. Future ballistic range testing might consider building test matrices to explicitly search for the equilibrium oscillation amplitude and constant energy amplitude growth behavior of the capsule. This would require greater control of initial conditions and more data per oscillation cycle than is typically gathered in spark-shadowgraph ranges. To find the amplitude of constant energy oscillation growth, high-speed movies or onboard telemetry would be required to measure the exact amplitude growth over a few cycles. Direct measurement of the oscillation amplitude is key for the equilibrium amplitude as well. Measuring free-flight data at a sufficient rate and with sufficient fidelity would be difficult, and many shots would be required to hunt for these characteristic features. However, designing tests to accurately measure amplitude growth, or for now, looking at available data with these features in mind might help to better define the functional forms of dynamic stability curves. More accurate functional forms would greatly facilitate all parameter identification techniques for determining blunt body capsule damping.

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